

K_2 of localisations of local rings

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Abstract

We show that K_2 of “sufficiently regular” localisations of local rings (e.g. inverting a sequence of regular parameters) can be described by the Steinberg presentation. The proof is inductive on the number of irreducible elements being inverted, successively using a generalisation of a co-Cartesian square first exploited by Dennis and Stein.

1 INTRODUCTION

The culmination of work by R. Dennis, M. Stein, W. van der Kallen, et al. [2, 3, 11, 12, 15, 17, 18] in the early 1970s showed that K_2 of a local ring A may be described via the Steinberg presentation (assuming that the residue field of A has > 5 elements):

$$K_2(A) = A^\times \otimes_{\mathbb{Z}} A^\times / \langle a \otimes 1 - a \mid a, 1 - a \in A^\times \rangle.$$

In other words, the natural map

$$K_2^M(A) \rightarrow K_2(A) \tag{*}$$

is an isomorphism, where K_*^M denotes Milnor K -theory. Their results apply more generally to 5-fold stable rings. The main goal of this paper is to show that the isomorphism (*) continues to hold if A is regular and we invert a suitable collection of elements of A :

Theorem 1.1. *Let A be a regular local ring whose residue field has > 5 elements, and let $t_1, \dots, t_n \in A$ be irreducible elements with the following property: the quotient of A by an ideal generated by any number of t_1, \dots, t_n is still regular. Then*

$$K_2^M(A_{t_1 \dots t_n}) \rightarrow K_2(A_{t_1 \dots t_n})$$

is an isomorphism.

Note that the ring $A_{t_1 \dots t_n}$ is not 5-fold stable (we check this in remark 3.5) and so the classical comparison results do not apply. In fact, as far as the author is aware, this is the first result showing that the Steinberg presentation remains valid for a class of rings outside the 5-fold stable ones (and arbitrary fields).

The idea of proof behind the main theorem is an induction in which we successively invert t_1, \dots, t_n using localisation sequences for both Milnor and Quillen K -theory in low degrees. Owing to the appearance of K_1 terms in the localisation sequences, we must first

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consider an analogue of the main theorem for K_1 , which is equivalent to giving conditions under which SK_1 of a localisation of a local ring vanishes; this is the purpose of section 2. In fact, using only this straightforward result for K_1 , we show in remark 2.6 that, in the notation of the main theorem, $K_2^M(A_{t_1 \dots t_n}) \rightarrow K_2(A_{t_1 \dots t_n})$ is surjective after tensoring by \mathbb{Q} .

Section 3 is then a summary of the classical work by Dennis, Stein, van der Kallen, et al., where we review the notion of k -fold stability, describe symbolic elements of K_2 , and give the basic properties the ‘‘Dennis–Stein–Suslin–Yarosh’’ ρ map. A version of this map was first used by Dennis and Stein [12] in their work proving that K_2 of a discrete valuation ring embeds into K_2 of its field of fractions; it was later extended to higher degree Milnor K -theory by S. Suslin and V. Yarosh [13].

Section 4 is the bulk of the proof of the main theorem. The first goal is to establish proposition 4.2, a co-Cartesian square generalising ones found in [4] and [13]. A corollary of this is a localisation sequence for Milnor K -theory. In the case of inverting a single prime element of A , the main theorem follows almost immediately by comparing this localisation sequence with the one for Quillen K -theory. In the general case, to proceed inductively, one must carefully check that, even though $A_{t_1 \dots t_{n-1}}$ is no longer 5-fold stable, it continues to satisfy certain similar properties. Most important of these is perhaps (A3) (immediately preceding proposition 4.2), stating that the group of units of $A_{t_1 \dots t_{n-1}}$ which are congruent to 1 mod t_n is generated by its elements of the form $1 + ut_n$ with $u \in A_{t_1 \dots t_{n-1}}^\times$.

Throughout the paper we give examples to show that certain assumptions cannot be discarded.

Example 1.2. We finish the introduction by providing a collection of examples to which our main theorem applies:

- (i) Let A be a regular, local ring whose residue field has > 5 elements, and let t_1, \dots, t_n be part of a sequence of regular parameters. Then the theorem holds.
- (ii) Let k be a field with > 5 elements and let A be the localisation of $k[X, Y]$ at the origin. Let $t_1, \dots, t_n \in A$ be local equations of lines through the origin (i.e. each t_i is a non-zero linear expression in X and Y). Then the theorem applies (and, by passing to the limit, the conclusion of the theorem would remain valid if we inverted the local equations of *all* lines through the origin).
- (iii) Let A be a regular, local ring whose residue field has > 5 elements. Then the ring $A[[t]]$ of formal Taylor series is still such a ring, and $A[[t], t]$ satisfies the conditions of the theorem. Therefore $K_2(A((t))) = K_2^M(A((t)))$, where $A((t)) = A[[t]][t^{-1}]$ is the ring of formal Laurent series.

NOTATION AND CONVENTIONS

All rings are commutative, unital, and Noetherian. We say that an element t of a ring R is a prime element if and only if it generates a non-zero, proper prime ideal; in a UFD this is the same as an irreducible element. When t is a prime element of a domain R , the localisation R_{tR} is a discrete valuation ring and we write ν_t for the associated t -adic valuation.

2 PRELIMINARY RESULTS ON K_0 AND K_1

We begin with some straightforward results on K_0 and K_1 . The following is classical but I did not know of a reference until the anonymous referee indicated [1, Rmk. 10.2]:

Lemma 2.1. *Let R be a regular ring for which $K_0(R) = \mathbb{Z}$. Then $K_0(S^{-1}R) = \mathbb{Z}$ for any multiplicative system $S \subset R$.*

Proof. Let V be a finitely generated projective module over $S^{-1}R$. Pick a finitely generated R module M for which $S^{-1}M = V$; since R is regular, M admits a finite length resolution by finitely generated projective R -modules. Each of these projectives is stably free by assumption, and so it is easy to modify the finite projective resolution to obtain a finite free resolution. Base changing to $S^{-1}R$ provides a finite free resolution of V , whence it is stably free. \square

Remark 2.2. Suppose that R is a domain and that $t \in R$ is a prime element. In section 4 we will be interested in the following assumption:

(A1) $R_t \cap R_{tR} = R$.

Assumption (A1) implies that $R_t^\times \cap R_{tR}^\times = R^\times$, and so the t -adic valuation induces an exact sequence

$$0 \rightarrow R^\times \rightarrow R_t^\times \rightarrow \mathbb{Z} \rightarrow 0.$$

In other words, a unit of R_t may be decomposed as ut^n for some unique $u \in R^\times$ and $n \in \mathbb{Z}$.

Assumption (A1) holds as soon as R is normal; indeed, in that case, $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$ (where \mathfrak{p} runs over the height one prime ideals of R) and $R_t \subseteq \bigcap_{\mathfrak{p} \neq tR} R_{\mathfrak{p}}$.

Proposition 2.3. *Suppose that R is a ring and that $t_1, \dots, t_n \in R$ is a sequence of non-zero-divisor, non-unit elements such that:*

- (i) R is regular, $K_0(R) = \mathbb{Z}$, and $K_1(R) \cong R^\times$.
- (ii) For $i = 1, \dots, n$, the ring R/t_iR is regular and $K_0(R/t_iR) = \mathbb{Z}$.

Then $K_1(R_{t_1 \dots t_n}) = R_{t_1 \dots t_n}^\times$.

Proof. Suppose first that $n = 1$; write $t = t_1$. Since R/tR is a regular ring whose spectrum is connected (since $K_0(R/tR) = \mathbb{Z}$), it is a domain. Therefore t is a prime element of R (which is a domain by the same argument); by the previous remark, there is an exact sequence

$$0 \rightarrow R^\times \rightarrow R_t^\times \rightarrow \mathbb{Z} \rightarrow 0.$$

Next consider the end of the long exact localisation sequence for K -theory:

$$K_1(R) \rightarrow K_1(R_t) \rightarrow K_0(R/tR) \rightarrow K_0(R) \rightarrow K_0(R_t).$$

Since $K_0(R) = \mathbb{Z}$, the rightmost arrow is injective (even an isomorphism by lemma 2.1), and so $K_1(R_t) \rightarrow K_0(R/tR)$ is surjective. Comparing our two sequences via the determinant and rank maps yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^\times & \longrightarrow & R_t^\times & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & K_1(R) & \longrightarrow & K_1(R_t) & \longrightarrow & K_0(R/tR) \longrightarrow 0 \end{array}$$

By assumption the left and right vertical arrows are isomorphisms, whence the central arrow is as well.

Now consider the general case, by induction. Put $R' = R_{t_1 \dots t_{n-1}}$ and $t = t_n$; so the inductive hypothesis implies $K_1(R') \cong R'^{\times}$.

Case: $t_i \in tR$ for some $i = 1, \dots, n-1$. Then t is already a unit in R' and so $R' = R_{t_1 \dots t_n}$; there is nothing more to show in this case.

Case: $t_i \notin tR$ for any $i = 1, \dots, n-1$. Since R/tR is a regular ring whose spectrum is connected, it is a domain; therefore $t_1 \dots t_{n-1}$ is non-zero in R/tR . So R' and R'/tR' are both localisations of regular rings with $K_0 = \mathbb{Z}$ by non-zero elements; therefore they are both themselves regular rings with $K_0 = \mathbb{Z}$, by lemma 2.1. This reduces the question to the case $n = 1$, which we have already treated. \square

The following corollary is the K_1 analogue of our main theorem:

Corollary 2.4. *Let A be a regular local ring and let $t_1, \dots, t_n \in A$ be non-zero elements such that $A/t_i A$ is regular for each i . Then $K_1(A_{t_1 \dots t_n}) = A_{t_1 \dots t_n}^{\times}$.*

Proof. This follows from the proposition since, for any local ring, $K_0 = \mathbb{Z}$ and $SK_1 = 0$. \square

Remark 2.5. To show the necessity of the regularity hypotheses, let A be a regular local ring and let $t \in A$. Arguing as in the proof of the previous proposition yields a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{\times} & \longrightarrow & A_t^{\times} & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \cong \uparrow & & \uparrow & & \uparrow \\ & & K_1(A) & \longrightarrow & K_1(A_t) & \longrightarrow & G_0(A/tA) \longrightarrow 0 \end{array}$$

If A/tA is not regular then there is a finitely generated A/tA module which has infinite projective dimension (e.g. the residue field); then $G_0(A/tA) \rightarrow \mathbb{Z}$ is not an isomorphism, and so $K_1(A_t) \rightarrow A_t^{\times}$ is also not an isomorphism.

Remark 2.6. From the corollary it is easy to deduce a statement weaker than the main theorem: If A, t_1, \dots, t_n satisfy the conditions of the main theorem, and if A is also assumed to contain a field, then $K_2(A_{t_1 \dots t_n})_{\mathbb{Q}}$ is generated by Steinberg symbols. In other words, $K_2(A_{t_1 \dots t_n})_{\mathbb{Q}} = K_2^{(2)}(A_{t_1 \dots t_n})_{\mathbb{Q}}$.

If $n = 0$ then this follows from theorem 3.7 below. Then we proceed by induction; put $R = A_{t_1 \dots t_{n-1}}$ and $t = t_n$. If $t \in R^{\times}$ there is nothing more to show, so suppose not. Consider the localisation sequence for $R \rightarrow R_t$, as well as its restriction to part of the Adams decomposition:

$$\begin{array}{ccccccc} K_2(R)_{\mathbb{Q}} & \longrightarrow & K_2(R_t)_{\mathbb{Q}} & \longrightarrow & K_1(R/tR)_{\mathbb{Q}} & & \\ \uparrow & & \uparrow & & \uparrow & & \\ K_2^{(2)}(R)_{\mathbb{Q}} & \longrightarrow & K_2^{(2)}(R_t)_{\mathbb{Q}} & \longrightarrow & K_1^{(1)}(R/tR)_{\mathbb{Q}} & \longrightarrow & K_1^{(2)}(R)_{\mathbb{Q}} \end{array}$$

(The fact that the localisation sequence decomposes into Adams summands in the indicated way is part of the Riemann–Roch theorem for higher algebraic K -theory, as least when A is smooth over a field; see [10, Thm. 7]. The more general case of A simply containing a field

then follows from Zariski descent for K -theory [14] and Neron–Popescu desingularisation [8, 9]). The previous corollary may be applied to A/tA , with elements $t_1, \dots, t_{n-1} \bmod tA$, to deduce that $K_1(R/tR) = (R/tR)^\times$; i.e. the right vertical map is an isomorphism. Secondly, the previous corollary also implies that $K_1(R) = R^\times$ and so $K_1^{(2)}(R)_\mathbb{Q} = 0$; therefore the central map on the bottom of the diagram is surjective. By the inductive hypothesis, the left vertical map is an isomorphism.

From a diagram chase we now see that the central vertical map is an isomorphism (one only needs to check it is surjective) completing the proof.

3 SYMBOLIC DESCRIPTIONS OF K_2

Here we review basic properties of K_2 and its presentations by Steinberg and Dennis–Stein symbols, as well as discussing the Dennis–Stein–Suslin–Yarosh ρ map.

3.1 RINGS WITH LOTS OF UNITS

There are various notions of when a ring has a lot of units; we will only need the following, which is a classical condition under which “general-position” type arguments work well:

Definition 3.1. Let $k \geq 1$ be an integer. A ring R is said to be *k -fold stable* if and only if whenever $a_1, b_1, \dots, a_k, b_k \in R$ are given such that $\langle a_1, b_1 \rangle = \dots = \langle a_k, b_k \rangle = R$, then there exists $r \in R$ such that $a_1 + rb_1, \dots, a_k + rb_k$ are units.

We will say that R is *weakly k -fold stable* if and only if whenever $b_1, \dots, b_{k-1} \in R$ are given, then there exists $u \in R^\times$ such that $1 + ub_1, \dots, 1 + ub_{k-1}$ are units. Note that weak k -fold stability follows from k -fold stability (using the pairs $\langle 1, b_1 \rangle, \dots, \langle 1, b_{k-1} \rangle, \langle 0, 1 \rangle$).

Remark 3.2. In other words, R is 1-fold stable if and only if $R^\times \rightarrow (R/aR)^\times$ is surjective for all $a \in R$.

Remark 3.3. A semi-local ring is k -fold stable if and only if all of its residue fields have $> k$ elements.

The notion of weak k -fold stability does not appear anywhere in the literature; we introduce it only to be able to clearly state the following and lemma 3.6:

Lemma 3.4. *If R is weakly k -fold stable then so is $S^{-1}R$ for any multiplicative system $S \subset R$.*

Proof. Let $b_1, \dots, b_{k-1} \in S^{-1}R$ be given, and pick $s \in S$ such that each b_i may be written as $b_i = b'_i s^{-1}$ for some $b'_i \in R$. By the weak k -fold stability of R , there exists $u \in R^\times$ such that $1 + ub'_1, \dots, 1 + ub'_{k-1} \in R^\times$. Replace u by us to complete the proof. \square

Our main result would follow from theorem 3.7 if it were the case that localisations of k -fold stable rings remained k -fold stable. The following example is provided to show that this is not the case; although we work with a specific example, the proof works in general:

Remark 3.5. Let \mathcal{O} be a discrete valuation ring with residue field K , and put $R = \mathcal{O}[[t]]$. Then R is k -fold stable for every $k < |K|$, but we will show that $R_t = \mathcal{O}((t))$ is not even 1-fold stable.

Let $\pi \in \mathcal{O}$ be a uniformiser. Since $\langle \pi, 1 + \pi^2 t^{-1} \rangle = \mathcal{O}((t))$, it is sufficient to show that there exists no $c \in \mathcal{O}((t))$ satisfying $\pi + c(1 + \pi^2 t^{-1}) \in \mathcal{O}((t))^\times$. For a contradiction, suppose such a c were to exist, and write $c = t^n c_0$ with $n \in \mathbb{Z}$, and $c_0 \in \mathcal{O}[[t]]$ not divisible by t . So,

$$v := \pi + t^n c_0(1 + \pi^2 t^{-1}) \in \mathcal{O}((t))^\times.$$

Write ν_t for the t -adic valuation on $\text{Frac}(\mathcal{O}[[t]])$; note that $vt^{-\nu_t(v)} \in \mathcal{O}[[t]]^\times$ by remark 2.2.

Case: $n \leq 0$. Then $\nu_t(v) = n - 1$, so $vt^{1-n} \in \mathcal{O}[[t]]^\times$. But

$$vt^{1-n} = \pi t^{1-n} + c_0(t + \pi^2),$$

which belongs to the maximal ideal of $\mathcal{O}[[t]]$ so is certainly not a unit.

Case: $n \geq 1$. Then $\nu_t(v) = 0$, so $v \in \mathcal{O}[[t]]^\times$, which is again absurd.

This completes the proof that \mathcal{O} is not even 1-fold stable. Yet, if $|K| > 5$, then our main theorem will imply that $K_2(\mathcal{O}((t)))$ is described by the Steinberg presentation.

3.2 STEINBERG SYMBOLS

Given $a, b \in R^\times$, the corresponding Steinberg symbol of $K_2(R)$ will be written $\{a, b\}$ as usual. These symbols [7, §8] satisfy the following relations in $K_2(R)$:

(S1) Bilinearity: $\{a, bc\} = \{a, b\} + \{a, c\}$ and $\{ac, b\} = \{a, b\} + \{c, b\}$ for $a, b, c \in R^\times$.

(S2) Skew-symmetry: $\{a, -a\} = 0$ for $a \in R^\times$.

(S3) Steinberg relation: $\{a, 1 - a\} = 0$ for $a \in R^\times$ such that $1 - a \in R^\times$.

Let $K_2^M(R)$ be the abelian group generated by symbols $\{a, b\}$, for $a, b \in R^\times$, subject to relations (S1) and (S3) (we will say a word about (S2) in a moment); this is the second Milnor K -group. Thus there is a natural homomorphism of abelian groups $K_2^M(R) \rightarrow K_2(R)$, $\{a, b\} \mapsto \{a, b\}$, whose image is the subgroup of $K_2(R)$ generated by the Steinberg symbols.

Lemma 3.6. *Suppose R is weakly 5-fold stable (e.g. a localisation of a local ring with > 5 elements in the residue field). Then relation (S2) follows from (S1) and (S3).*

Proof. Suppose $a \in R^\times$; we must show that $\{a, -a\} = 0$ in $K_2^M(R)$. The proof is well-known (e.g. [5]), but it is important to take care to use only the weak version of k -fold stability:

Firstly, if actually $1 - a \in R^\times$, then use the identity $-a(1 - a^{-1}) = 1 - a$ to deduce that $\{a, -a\} = \{a, 1 - a\} - \{a, 1 - a^{-1}\} = 0$.

Secondly suppose that $s \in R^\times$ happens to satisfy $1 - s, 1 - sa \in R^\times$; then the first part of the proof tells us that $\{as, -as\} = \{s, -s\} = 0$. Therefore,

$$\begin{aligned} 0 &= \{as, -as\} \\ &= \{a, -a\} + \{a, -s\} + \{s, -a\} + \{s, -s\} + \{a, -1\} + \{s, -1\} \end{aligned}$$

i.e. $\{a, -a\} = -\{a, s\} - \{s, a\}$.

Thirdly, apply weak 3-fold stability to $-1, -a$ to see that there exists an element $s_1 \in R^\times$ with the above properties. Next apply weak 5-fold stability to $-1, -a, -s_1, -s_1 a$ to find

$s_2 \in R^\times$ which not only satisfies the above properties, but such that $s = s_1 s_2$ also satisfies the above properties. So,

$$\{a, -a\} = -\{a, s_1\} - \{s_1, a\},$$

$$\{a, -a\} = -\{a, s_2\} - \{s_2, a\},$$

and

$$\{a, -a\} = -\{a, s_1 s_2\} - \{s_1 s_2, a\}.$$

It follows that $\{a, -a\} = 0$. □

The main classical theorem concerning the presentation of K_2 by Steinberg symbols is the following, due to W. van der Kallen, H. Maazen, and J. Stienstra [17, §8] [18]:

Theorem 3.7. *If R is 5-fold stable, then $K_2(R)$ is generated by the Steinberg symbols subject to relations (S1) and (S3); in other words, $K_2^M(R) \rightarrow K_2(R)$ is an isomorphism.*

Remark 3.8. We will never need the proper definition of the Steinberg symbol, but include it here for completeness. Recall that $K_2(R)$ may be defined as

$$K_2(R) = \text{Ker}(St(R) \longrightarrow E(R)),$$

where $St(R)$ is the Steinberg group and $E(R)$ is the group of infinite elementary matrices over R . The typical generators of $St(R)$ are

$$x_{ij}(a) \quad (a \in R, i, j \geq 0, i \neq j).$$

Put $w_{ij}(a) = x_{ij}(a)x_{ji}(-a^{-1})x_{ij}(a)$ and $h_{ij}(a) = w_{ij}(a)w_{ij}(-1)$.

In this notation, the Steinberg symbol is given by

$$\{a, b\} := [h_{12}(a), h_{13}(b)] \in K_2(R)$$

whenever $a, b \in R^\times$.

We are following the modern convention of writing $K_2(R)$ additively, especially when performing symbolic manipulations in it, even though it is a subgroup of the non-abelian group $St(R)$ where we must use multiplicative notation; this should not lead to confusion since quite soon all manipulations will be via symbols.

3.3 DENNIS–STEIN SYMBOLS

Given $a, b \in R$ such that $1 + ab \in R^\times$, Dennis and Stein [12] defined an element of $K_2(R)$ denoted $\langle a, b \rangle$. These symbols satisfy the following:

(D1) $\langle a, b \rangle = -\langle -b, -a \rangle$ for $a, b \in R$ such that $1 + ab \in R^\times$.

(D2) $\langle a, b \rangle + \langle a, c \rangle = \langle a, b + c + abc \rangle$ for $a, b, c \in R$ such that $1 + ab, 1 + ac \in R^\times$.

(D3) $\langle a, bc \rangle = \langle ab, c \rangle + \langle ac, b \rangle$ for $a, b, c \in R$ such that $1 + abc \in R^\times$.

Moreover, any Steinberg symbol $\{a, b\}$, where $a, b \in R^\times$, is equal to a Dennis–Stein symbol in $K_2(R)$:

$$\{a, b\} = \langle (a-1)b^{-1}, b \rangle.$$

It follows that the Dennis–Stein symbol $\langle a, b \rangle$ can be expressed as a Steinberg symbol whenever a or b is a unit:

$$\langle a, b \rangle = \begin{cases} \{-a, 1+ab\} & \text{if } a \in R^\times, \\ \{1+ab, b\} & \text{if } b \in R^\times. \end{cases}$$

The main classical theorem concerning the presentation of K_2 by Dennis–Stein symbols is the following, again due to W. van der Kallen, H. Maazen, and J. Stienstra [17, 18]:

Theorem 3.9. *If R is local or 3-fold stable, then $K_2(R)$ is the abelian group generated by the Dennis–Stein symbols subject to relations (D1)–(D3).*

Remark 3.10. To be precise, the Dennis–Stein symbol is defined to be

$$\langle a, b \rangle = x_{21} \left(\frac{-b}{1+ab} \right) x_{12}(a) x_{21}(b) x_{12} \left(\frac{-a}{1+ab} \right) h_{12}(1+ab)^{-1} \in K_2(R),$$

though we will never need this formula.

Some later literature, e.g. [6], prefers to denote by $\langle a, b \rangle$ our $\langle -a, b \rangle$.

3.4 THE DENNIS–STEIN–SUSLIN–YAROSH ρ MAP

Dennis and Stein proved in [4] that if \mathcal{O}_K is a discrete valuation ring with field of fractions K , then $K_2(\mathcal{O}_K) \rightarrow K_2(K)$ is injective. They did this by constructing a certain homomorphism $\rho : 1 + t\mathcal{O}_K \rightarrow K_2(\mathcal{O}_K)$ (where $t \in \mathcal{O}_K$ is a uniformiser) such that the image of $\rho(u)$ in $K_2(K)$ is $\{u, t\}$. The map was extended to the higher degree Milnor K -theory of \mathcal{O}_K by Suslin and Yarosh in [13].

Dennis and Stein required rather intricate usage of their so-called (s, t) -identities between Steinberg symbols (for example, to prove the following two results) since, at the time, they did not know that they could describe $K_2(\mathcal{O}_K)$ using their Dennis–Stein symbols. With this hindsight, it is not difficult to give a much easier and more general definition of their ρ map; the payoff is the co-Cartesian square of proposition 4.2, a version of which appeared in their work.

Let R be a ring. For $t \in R$ a non-zero-divisor, write $(1+tR)^\times = (1+tR) \cap R^\times$, i.e. the elements of $1+tR$ which are units in R ; note that $(1+tR)^\times$ is a group. Define

$$\rho_t : (1+tR)^\times \rightarrow K_2(R), \quad 1+ta \mapsto \langle a, t \rangle,$$

where the right hand side is a Dennis–Stein symbol.

Lemma 3.11. *ρ_t is a homomorphism of groups. Moreover, if $s \in R$ is another non-zero-divisor and $f \in (1+stR)^\times$, then*

$$\rho_{st}(f) = \rho_s(f) + \rho_t(f).$$

Proof. Relations (D1) and (D2) show that ρ_t is a homomorphism. Writing $f = 1 + sta$ for some $a \in R$, the second claim is equivalent to

$$\langle a, st \rangle = \langle as, t \rangle + \langle at, s \rangle,$$

which is immediate from relation (D3). \square

Corollary 3.12. *Suppose $u \in R^\times$ is such that $1 + t^l u \in R^\times$ for some $l \geq 1$; then*

$$l\rho_t(1 + t^l u) = \{-u, 1 + t^l u\}$$

in $K_2(R)$.

Proof. By the previous lemma,

$$l\rho_t(1 + t^l u) = \rho_{t^l}(1 + t^l u) = \langle u, t^l \rangle,$$

which equals $\{-u, 1 + t^l u\}$ by the comparison between Dennis–Stein and Steinberg symbols. \square

Remark 3.13. In particular, the previous corollary in the case $l = 1$ states that if $u \in R^\times$ satisfies $1 + tu \in R^\times$, then

$$\rho_t(1 + tu) = \{-u, 1 + tu\}.$$

Suppose that $(1 + tR)^\times$ is generated by its elements of the form $1 + tu$ with $u \in R^\times$. For example, this is true if R is 3-fold stable (see (A3) in section 4), or if R is local and $t \in \mathfrak{m}_R$. Then it follows that the homomorphism $\rho_t : (1 + tR)^\times \rightarrow K_2(R)$ is characterised by the property that $\rho_t(1 + tu) = \{-u, 1 + tu\}$ for all $u \in R^\times$ for which $1 + tu \in R^\times$.

Remark 3.14. Suppose that a, t are elements of a ring R such that $1 + a, 1 - t, 1 + at$ are all in R^\times ; then

$$\langle a, t \rangle = \left\{ -\frac{1+a}{1-t}, \frac{1+at}{1-t} \right\}.$$

This is proved by manipulating Dennis–Stein symbols as follows: Firstly, the comparison between Steinberg and Denis–Stein symbols implies $\left\{ -\frac{1+a}{1-t}, \frac{1+at}{1-t} \right\} = \langle \frac{1+a}{1-t}, t \rangle$. Next, relation (D3) implies $\langle -t, -\frac{1+a}{1-t} \rangle + \langle -t, 1 \rangle = \langle -t, -a \rangle$, which (D1) converts to $\langle \frac{1+a}{1-t}, t \rangle = \langle a, t \rangle + \langle -t, 1 \rangle$. Finally note that $\langle -1, t \rangle = 0$: indeed, for any r such that $1 + r \in R^\times$, relation (D3) gives $\langle r, 1 \rangle = \langle r, 1 \rangle + \langle r, 1 \rangle$, whence $\langle r, 1 \rangle = 0$. This completes the proof of the claimed identity.

Therefore, if R is a local ring and $t \in \mathfrak{m}_R$ is a non-zero-divisor, we conclude that

$$\rho_t(1 + at) = \begin{cases} \{-a, 1 + at\} & \text{if } a \in R^\times, \\ \left\{ -\frac{1+a}{1-t}, \frac{1+at}{1-t} \right\} & \text{if } a \in \mathfrak{m}_R. \end{cases}$$

In the case of a discrete valuation ring with t being a uniformiser, this was Dennis and Stein’s original definition of their map $\rho = \rho_t$, and they then proved directly, using their so-called (s,t)-identities, that it is a homomorphism.

4 THE MAIN CALCULATIONS

In this section we prove the main result. First we will construct, under a number of assumptions, a short exact sequence (see corollary 4.3), which serves as a K_2^M -analogue of the sequence in remark 2.2. From this, the proof that Milnor and Quillen K_2 coincide will proceed by a slightly tricky induction; to keep this as clear as possible, we carefully label each assumption as it appears.

Let R be a domain and suppose that $t \in R$ is a prime element of R . Write $\nu = \nu_t$ for the t -adic discrete valuation on $\text{Frac } R$, with ring of integers R_{tR} . To start, we make the following assumption, which we briefly discussed in remark 2.2:

(A1) Assume that $R_t \cap R_{tR} = R$. (E.g. This is true if R is normal; see the aforementioned remark.)

The assumption implies that any unit f in R_t may be written as $f = ut^n$ for some unique $u \in R^\times$ and $n \in \mathbb{Z}$; indeed, $n = \nu(f)$ and $u = ft^{-\nu(f)}$. Supposing that $g = vt^m$ is another unit of R_t , written in the same way, the tame symbol $c(f, g)$ is defined in the usual way as

$$c(f, g) := (-1)^{\nu(f)\nu(g)} f^{\nu(g)} g^{-\nu(f)} = (-1)^{nm} u^m v^{-n} \in R^\times.$$

Lemma 4.1. *Let R, t be as above, satisfying (A1). If $f, g \in R_t^\times$ satisfy $f + g = 1$, then $c(f, g) \in (1 + tR)^\times$ and*

$$\rho_t(c(f, g)) = -\{ft^{-\nu(f)}, gt^{-\nu(g)}\}$$

in $K_2(R)$, where the left side is the Dennis–Stein–Suslin–Yarosh map from section 3.4.

Proof. Write $f = ut^n$, $g = vt^m$ as above; we must show that $c(f, g) \in 1 + tR$ (we already know it is a unit) and that $\rho_t(c(f, g)) = -\{u, v\}$. The proof is a simple case-by-case analysis:

Case: $m > 0$. Then $n = 0$ and $u = 1 - vt^m$. So $c(f, g) = u^m \in 1 + t^m R$ and

$$\rho_t(c(f, g)) = m\rho_t(1 - vt^m)$$

as ρ_t is a homomorphism (lemma 3.11). But corollary 3.12 implies $m\rho_t(1 - vt^m) = \{v, 1 - vt^m\} = \{v, u\}$, as required.

Case: $n > 0$. Proceed as in the previous case.

Case: $n = m = 0$. Then $u + v = 1$, whence $\{u, v\} = 0$, and secondly $c(f, g) = 1$. So the claim is trivial.

Case: $m < 0$. Then $n = m$ (set $l = -n$ for clarity), $u + v = t^l$, and

$$c(f, g) = (-1)^l u^{-l} v^l = (1 - u^{-1}t^l)^l \in 1 + t^l R.$$

Using the fact that ρ_t is a homomorphism and corollary 3.12, we see that

$$\rho_t(c(f, g)) = l\rho_t(1 - u^{-1}t^l) = \{u^{-1}, 1 - u^{-1}t^l\} = \{u^{-1}, -vu^{-1}\}.$$

But this symbol equals $\{u^{-1}, v\} = -\{u, v\}$, as required. □

Before the next proposition, we must impose another two assumptions:

- (A2) Assume that R is weakly 5-fold stable and that $K_2^M(R) \rightarrow K_2(R)$ is an isomorphism. (E.g. This is true if R is 5-fold stable.)
- (A3) Assume that the group $(1 + tR)^\times$ is generated by its elements $1 + tw$ satisfying the extra condition that $w \in R^\times$. (E.g. This is true if R is 3-fold stable: Let $a \in R$ be such that $1 + at$ is a unit. Applying 3-fold stability to $\langle a, -1 \rangle, \langle 1, t \rangle, \langle 0, 1 \rangle$ supplies us with a unit $v \in R^\times$ for which $a - v$ and $1 + vt$ are also units. Put $w = \frac{a-v}{1+vt} \in R^\times$. Then $(1 + vt)(1 + wt) = 1 + at$, which implies that $1 + wt$ is a unit and completes the proof.)

Under assumption (A2) we freely identify the two K_2 -groups and even allow ourselves to think of Dennis–Stein symbols as elements of $K_2^M(R)$; this assumption also implies that R_t is weakly 5-fold stable (lemma 3.4) and therefore that $K_2^M(R_t)$ satisfies skew-symmetry (lemma 3.6), which we will use without mention.

Proposition 4.2 (c.f. [4] [13]). *Let R, t be as above, satisfying (A1) – (A3). Then*

$$\begin{array}{ccc} (1 + tR)^\times & \xrightarrow{j} & R^\times \\ \rho_t \downarrow & & \downarrow \{\cdot, t\} \\ K_2^M(R) & \longrightarrow & K_2^M(R_t) \end{array}$$

is a co-Cartesian square of abelian groups.

Proof. Since both maps from $(1 + tR)^\times$ to $K_2^M(R_t)$ are homomorphisms, it is enough, by (A3), to check the commutativity of the diagram on elements of $(1 + tR)^\times$ having the form $1 + wt$ for some $w \in R^\times$. For such an element, $\rho_t(1 + tw) = \langle w, t \rangle = \{-w, 1 + wt\}$ in $K_2^M(R)$; the image of this in $K_2^M(R_t)$ is

$$\{-w, 1 + wt\} = \{-wt, 1 + wt\} - \{t, 1 + wt\} = 0 + \{1 + wt, t\},$$

as required.

Given $f = ut^n$, $g = vt^m \in R_t^\times$, written as above, we have, in $K_2^M(R_t)$,

$$\{f, g\} = \{u, v\} + \{u^m, t\} + \{t, v^n\} + \{t, t\} = \{u, v\} + \{c(f, g), t\}.$$

This shows that

$$K_2^M(R) \oplus R^\times \rightarrow K_2^M(R_t), \quad (\{u, v\}, c) \mapsto \{u, v\} + \{c, t\}$$

is surjective.

Let $\Delta : (1 + tR)^\times \rightarrow K_2^M(R) \oplus R^\times$ be the map $(-\rho_t, j)$, so that $X = (K_2^M(R) \oplus R^\times) / \text{Im } \Delta$ is the pushout which we wish to show is equal to $K_2^M(R_t)$. The previous lemma shows that the homomorphism

$$K_2^M(R_t) \rightarrow X, \quad \{f, g\} \mapsto (\{ft^{-\nu(f)}, gt^{-\nu(g)}\}, c(f, g)) \pmod{\text{Im } \Delta}$$

is well-defined. But we have just shown that the natural map $X \rightarrow K_2^M(R_t)$ is surjective, and the reader can easily check that $X \rightarrow K_2^M(R_t) \rightarrow X$ is the identity, thereby completing the proof. \square

Corollary 4.3. *Let R, t be as above, satisfying (A1) – (A3). Then there is a short exact sequence*

$$0 \rightarrow K_2^M(R) \rightarrow K_2^M(R_t) \xrightarrow{\bar{c}} R^\times / (1 + tR)^\times \rightarrow 0,$$

where $\bar{c}(\{f, g\}) := c(\{f, g\}) \bmod tR$.

Example 4.4. (i) Suppose that \mathcal{O}_K is a discrete valuation ring whose residue field k has > 5 elements; let K be its fraction field and $t \in \mathcal{O}_K$ a uniformiser. Then the pair \mathcal{O}_K, t satisfy assumptions (A1) – (A3) and, moreover, $K_2^M(K) \cong K_2(K)$ by Matsumoto's theorem [7, §12]. We deduce that

$$0 \rightarrow K_2(\mathcal{O}_K) \rightarrow K_2(K) \rightarrow k^\times \rightarrow 0$$

is exact, which is the main result of [4].

(ii) Suppose that A is a local domain whose residue field has > 5 elements. Then the pair $A[[t]], t$ satisfies (A1) – (A3) and so the sequence

$$0 \rightarrow K_2^M(A[[t]]) \rightarrow K_2^M(A((t))) \rightarrow A^\times \rightarrow 0$$

is exact.

(iii) Suppose that A is a normal, local ring whose residue field k has > 5 elements, and let $t \in A$ be any prime element. Then the pair A, t satisfies (A1) – (A3) and so the sequence

$$0 \rightarrow K_2^M(A) \rightarrow K_2^M(A_t) \rightarrow (A/tA)^\times \rightarrow 0$$

is exact.

The key inductive step of our main proof is contained in the following proposition, in which we give conditions under which the short exact sequence of the corollary forces $K_2^M(R_t) \rightarrow K_2(R_t)$ to be an isomorphism.

Proposition 4.5. *Let R, t be as above, satisfying (A1) – (A3). Suppose further that R and R/tR are regular, and that*

(A4) $R^\times \rightarrow (R/tR)^\times$ is surjective. (E.g. This is true if R is 1-fold stable by remark 3.2.)

(A5) $K_1(R/tR) = (R/tR)^\times$ and $K_2(R/tR) = K_2^M(R/tR)$.

Then $K_2^M(R_t) \rightarrow K_2(R_t)$ is an isomorphism.

Proof. Let G_* denote the K -theory of the category of finitely generated modules over a ring. The following standard argument [16] shows that $G_*(R) \rightarrow G_*(R/tR) \rightarrow G_*(R)$ is zero: it is represented by $-\otimes_R R/tR : R\text{-Mod} \rightarrow R\text{-Mod}$ and the class of R/tR in $G_0(R)$ is trivial. Thus $K_*(R) \rightarrow K_*(R/tR) \rightarrow K_*(R)$ is zero. Assumption (A4) and the first part of (A5) implies that $R^\times \rightarrow K_1(R) \rightarrow K_1(R/tR) = (R/tR)^\times$ is surjective, and so it follows that $K_1(R/tR) \rightarrow K_1(R)$ is zero. Similarly, assumption (A4) and the second part of (A5) imply that $K_2(R) \rightarrow K_2(R/tR)$ is surjective, and so $K_2(R/tR) \rightarrow K_2(R)$ is zero.

The localisation sequence for K -theory therefore produces the short exact sequence

$$0 \rightarrow K_2(R) \rightarrow K_2(R_t) \rightarrow K_1(R/tR) \rightarrow 0.$$

We compare this with the short exact sequence of the previous corollary to get the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_2^M(R) & \longrightarrow & K_2^M(R_t) & \longrightarrow & R^\times / (1 + tR)^\times \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K_2(R) & \longrightarrow & K_2(R_t) & \longrightarrow & K_1(R/tR) \longrightarrow 0
 \end{array}$$

The left vertical arrow is an isomorphism by (A2), and the right vertical arrow is an isomorphism by (A4) and the first part of (A5). The proof is complete. \square

Except for a small lemma which we defer for a moment, we have reached the main theorem:

Theorem 4.6. *Let A be a regular local ring whose residue field has > 5 elements, and let $t_1, \dots, t_n \in A$ be irreducible elements with the following property: the quotient of A by an ideal generated by any number of t_1, \dots, t_n is still regular. Then*

$$K_2^M(A_{t_1 \dots t_n}) \rightarrow K_2(A_{t_1 \dots t_n})$$

is an isomorphism.

Proof. To avoid confusing the exposition with the special case $n = 1$, we quickly deal with it now; put $t = t_1$ and note that A/tA is regular. We will be done as soon as we verify that A, t satisfy all the other conditions of the previous proposition: (A1) holds because A is regular, hence normal; (A2) holds because A is 5-fold stable; (A3) holds because A is 3-fold stable; (A4) holds because A is 1-fold stable; the first part of (A5) holds because A/tA is a local ring, and the second part holds because A/tA is 5-fold stable. This concludes the proof in the case $n = 1$.

The remainder of the proof is by induction on $n > 1$ using the previous results. We may obviously assume that the t_1, \dots, t_n are pairwise non-associated. Let $R = A_{t_1 \dots t_{n-1}}$ and put $t = t_n$. We will show that the pair R, t satisfies all the conditions of the previous proposition, from which the result then follows.

(A1): By the assumption that t_1, \dots, t_n were pairwise non-associated, we see that t is a prime element of R ; moreover, R is normal. So the pair R, t satisfy assumption (A1).

(A2): By the inductive hypothesis, $K_2^M(R) \rightarrow K_2(R)$ is an isomorphism. Moreover, R is weakly 5-fold stable by lemma 3.4. So the pair R, t satisfy (A2).

(A3): This will be covered by the next lemma.

(A4): Set $\bar{A} = A/tA$, $\bar{R} = R/tR$, and let \bar{t}_i denote the image of t_i in A/tA , for $i = 1, \dots, n - 1$; so $\bar{R} = \bar{A}_{\bar{t}_1 \dots \bar{t}_{n-1}}$. Then \bar{A} is regular. Also, for each $i = 1, \dots, n - 1$, the element \bar{t}_i is non-zero in \bar{A} and $\bar{A}/\langle \bar{t}_i \rangle = A/\langle t_i, t \rangle$ is a regular local ring, hence a domain; so \bar{t}_i is a prime element of \bar{A} . By repeatedly applying the comments immediately after the introduction of (A1) at the start of this section, we see that a unit w of \bar{R} may be written as $u \bar{t}_1^{\alpha_1} \dots \bar{t}_{n-1}^{\alpha_{n-1}}$ for some $u \in \bar{A}^\times$ and $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{Z}$ (the representation might not be unique because some of the $\bar{t}_1, \dots, \bar{t}_{n-1}$ may be associated to one another). Since A is local, there is a unit $\tilde{u} \in A^\times$ sitting over u ; then $\tilde{u} t_1^{\alpha_1} \dots t_{n-1}^{\alpha_{n-1}} \in R^\times$ sits over w . This proves that R, t satisfies (A4).

(A5) The quotient of \bar{A} by the ideal generated by any number of $\bar{t}_1, \dots, \bar{t}_{n-1}$ is still regular, so the inductive hypothesis implies that $K_2^M(\bar{R}) \rightarrow K_2(\bar{R})$ is an isomorphism, while corollary 2.4 implies that $K_1(\bar{R}) = \bar{R}^\times$. So condition (A5) is satisfied for the pair R, t .

Finally, note that R and $R/tR = \bar{R}$ are localisations of regular rings, hence are regular. So the pair R, t satisfy all the required conditions to apply the previous proposition, except possibly (A3), which is rather subtle and which we deal with in the next lemma. \square

Lemma 4.7. *Let A, t_1, \dots, t_n satisfy the conditions of the theorem, with t_1, \dots, t_n being pairwise non-associated; put $R = A_{t_1 \dots t_{n-1}}$ and $t = t_n$. Then the pair R, t satisfy assumption (A3).*

Proof. We have explained the case $n = 1$ several times already, so we assume that $n > 1$ and proceed inductively.

Suppose $a \in R$ is such that $1 + at \in R^\times$. If $a \in A$ then, as we noticed when introducing assumption (A3), the 3-fold stability of A supplies us with $v, w \in A^\times$ such that $1 + at = (1 + vt)(1 + wt)$ and $1 + vt \in A^\times$; so $1 + wt \in R^\times$ and this completes the proof in this case. It remains to treat the case that $a \notin A$. This means that a has a denominator containing at least one of t_1, \dots, t_{n-1} ; after reordering for simplicity, we assume t_{n-1} occurs in the denominator of a .

Put $R' = R_{t_1 \dots t_{n-2}}$ ($= A$ if $n = 2$), so that $R = R'_{t_{n-1}}$. We have arranged matters so that $a = t_{n-1}^{-\alpha} b$ for some $\alpha > 0$ and some $b \in R'$ which is not divisible by t_{n-1} . Put

$$u := t_{n-1}^\alpha + bt \in R^\times,$$

and note that u is also a unit in $R'_{t_{n-1}R'}$ (since $\alpha > 0$ and bt is not divisible by t_{n-1} in R'); as usual, since t_{n-1} is a prime element of R' , which is normal, this implies that $u \in R'^\times$.

It follows that $t_{n-1} \bmod tR'$ is a unit in R'/tR' . Exactly as we argued in the previous theorem to prove (A4), this means that $\bar{t}_{n-1} = u\bar{t}_1^{\alpha_1} \cdots \bar{t}_{n-2}^{\alpha_{n-2}}$ in R'/tR' for some $u \in (A/tA)^\times$ and $\alpha_1, \dots, \alpha_{n-2} \in \mathbb{Z}$, where we write $\bar{}$ to denote images mod t . But, also as we noted when proving (A4) above, $\bar{t}_1, \dots, \bar{t}_{n-1}$ are prime elements of A/tA ; hence \bar{t}_{n-1} is associated to \bar{t}_j for some $j = 1, \dots, n-2$.

Therefore there exists $c \in A$ and $w \in A^\times$ such that

$$t_{n-1} + ct = wt_j. \quad (\ddagger)$$

Examining this equation modulo t_{n-1} , and using that t, t_j are prime elements of $A/t_{n-1}A$, we deduce that $t \bmod t_{n-1}$ and $t_j \bmod t_{n-1}$ are associated and, more importantly, that $c \bmod t_{n-1}$ is a unit in $A/t_{n-1}A$. Since t_{n-1} is in the Jacobson radical of A , this implies c was already a unit of A . Having formula (\ddagger) and the knowledge that $c \in A^\times$, the rest of the proof is less obtuse.

From (\ddagger) and the formula for u , we have

$$\begin{aligned} 1 + at &= t_{n-1}^{-\alpha} u \\ &= \left(w^{-1} t_j^{-1} \left(1 + \frac{c}{t_{n-1}} t \right) \right)^\alpha u \\ &= \left(1 + \frac{c}{t_{n-1}} t \right)^\alpha v \end{aligned}$$

where $v := w^{-\alpha}ut_j^{-\alpha} \in R^\times$. So $1 + \frac{c}{t_{n-1}}t$ is also in R^\times , and we note that $\frac{c}{t_{n-1}} \in R^\times$. Moreover, it follows that $v \in (1 + tR)^\times$; hence $v \in (1 + tR)^\times \cap R^\times = (1 + tR')^\times$, and the inductive hypothesis completes the proof. \square

Remark 4.8. I do not know whether the previous lemma remains valid for more general localisations of local rings. It would seem to offer a useful tool in the study of Milnor K -theory, especially when combined with the Dennis–Stein–Suslin–Yarosh map.

Remark 4.9. To see the necessity of some regularity hypotheses for the main theorem to be valid, consider the following situation: let A be a local domain whose residue field has > 5 elements, and recall from example 4.4(ii) that the sequence

$$0 \rightarrow K_2^M(A[[t]]) \rightarrow K_2^M(A((t))) \rightarrow A^\times \rightarrow 0$$

is exact. There is an analogous complex in Quillen K -theory which results from the fundamental sequence for Laurent polynomials. Namely,

$$K_2(A[[t]]) \hookrightarrow K_2(A((t))) \twoheadrightarrow A^\times$$

is a complex and $K_2(A((t)))$ decomposes as a direct sum $K_2(A((t))) = K_2(A[[t]]) \oplus A^\times \oplus NK_2(A)$, where $NK_2(A) \cong \text{coker}(K_2(A) \rightarrow K_2(A[X]))$ (see [19] for a proof).

Comparing the sequences for Milnor and Quillen K_2 , we see that $K_2^M(A((t))) \rightarrow K_2(A((t)))$ is an isomorphism if and only if $NK_2(A) = 0$, which is true if A is regular but is not true in general.

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